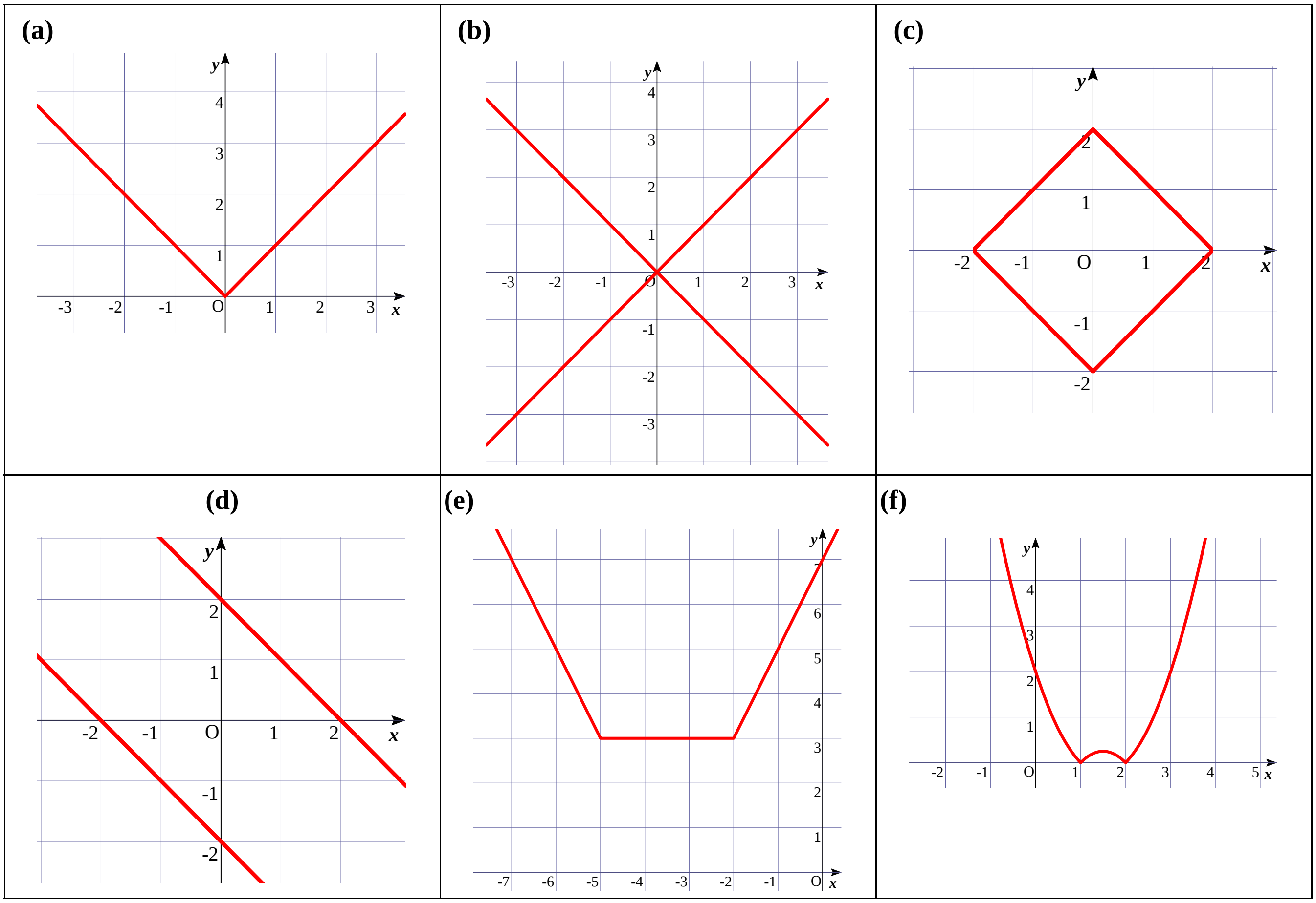
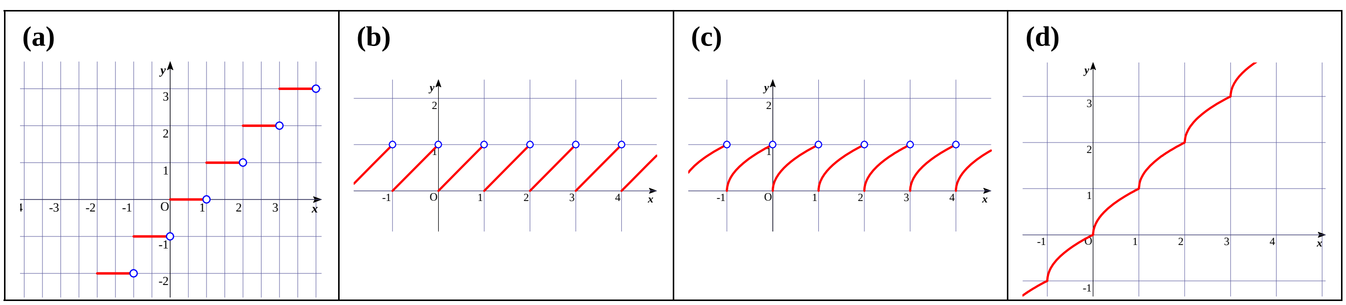
Functions and Graphs

1.



2.



3. **(a)** $G(x) = f(x) + f(-x)$, $G(-x) = f(-x) + f(x)$, $\therefore G(x) = G(-x)$ and $G(x)$ is even.

$H(x) = f(x) - f(-x)$, $H(-x) = f(-x) - f(x)$, $\therefore H(-x) = -H(x)$ and $H(x)$ is odd.

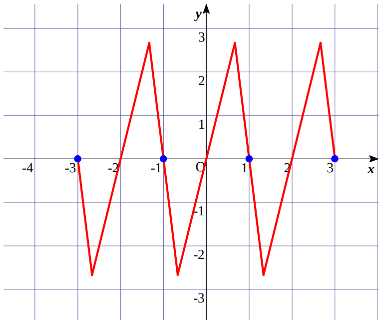
(b) $f(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2}$, which is the sum of an even and an odd function.

4. **(a)** $\sin^2(x+\pi) = (-\sin x)^2 = \sin^2 x$. $\therefore$ $\sin^2 x$ is periodic and the smallest period is π.

(b) $\sin(x^2)$ is not periodic.

(c) **(i)** 2π **(ii)** $\pi/2$

5.



6. **(a)** $f(x + y) = f(x)\, g(y) + g(x)\, f(y)$; $g(x + y) = g(x)\, g(y) - f(x)\, f(y)$

Put $x = y = 0$, $f(0) = 2f(0)g(0)$ …. (1) ; $g(0) = [g(0)]^2 - [f(0)]^2$ ….(2)

From (1), $f(0)\,[1 - 2g(0)] = 0 \Rightarrow f(0) = 0$ or $g(0) = 1/2$

If $g(0) = 1/2$, from (2), $1/2 = 1/4 - [f(0)]^2$, $[f(0)]^2 = -1/4$, which has no solution.

$\therefore$ $f(0) = 0$, then from (2), $g(0) = [g(0)]^2 \Rightarrow g(0) = 0$ or $g(0) = 1$

However, if $f(0) = 0$ and $g(0) = 0$, we have

$g(x) = g(x + 0) = g(x)\, g(0) + f(x)\, f(0) = 0$, hence $g(x) \equiv 0$, contradicts with assumption.

$\therefore$ $f(0) = 0$ and $g(0) = 1$

(b) $f'(0) = 1$, $g'(0) = 0$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)}{h}=1,\quad \lim_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim_{h\to 0}\frac{g(h)-1}{h}=0,\quad \lim_{h\to 0}\frac{g(h)}{h}=\lim_{h\to 0}\frac{1}{h}=0$$

(i) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)g(h)+g(x)f(h)}{h}=f(x)\lim_{h\to 0}\frac{g(h)}{h}+g(x)\lim_{h\to 0}\frac{f(h)}{h}$$

$= f(x) \times 0 + g(x) \times 1 = g(x)$

(ii) $$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{g(x)g(h)-f(x)f(h)}{h}=g(x)\lim_{h\to 0}\frac{g(h)}{h}-f(x)\lim_{h\to 0}\frac{f(h)}{h}$$

$= g(x) \times 0 - f(x) \times 1 = -f(x)$

(c) From (b), $f''(x) = g'(x) = -f(x)$, $\therefore f(x) = k_1 \cos x + k_2 \sin x$,

$f(0) = 0$, $f'(0) = 1 \Rightarrow k_1 = 0, k_2 = 1$ $\therefore f(x) = \sin x$

From (b), $g''(x) = -f'(x) = -g(x)$, $\therefore g(x) = k_3 \cos x + k_4 \sin x$,

$g(0) = 1$, $g'(1) = 0 \Rightarrow k_3 = 1, k_4 = 0$ $\therefore g(x) = \cos x$.

7. f, g are even $\Rightarrow f(-x) = f(x)$ and $g(-x) = g(x) \Rightarrow f(-x) + g(-x) = f(x) + g(x) \Rightarrow f + g$ is even.

Let $h(x) = \sum_{i=0}^{n} a_i x^{2i}$, then $h(-x) = \sum_{i=0}^{n} a_i (-x)^{2i} = \sum_{i=0}^{n} a_i x^{2i} = h(x)$, $\therefore \sum_{i=0}^{n} a_i x^{2i}$ is even.

8. **(a)** False, $f(x) = x$ is odd, $f(x) = |x|$.

(b) True, $h(x) = \frac{1}{2}[g(x)+g(-x)]$, $h(-x) = \frac{1}{2}[g(-x)+g(x)] = h(x)$, $\therefore h(x)$ is even.

(c) True, If f is injective, then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

If f is even, then $f(-x) = f(x)$, but by the injective property, we have $-x = x$, $x = 0$.

This leads to contradiction unless the domain of $f = \{0\}$.

(d) True, If f is increasing, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

$\therefore$ If $x > 0$, we have $f(-x) < f(x)$. If $x < 0$, we have $f(-x) > f(x)$.

In both cases, we don't have $f(-x) = f(x)$ and f is not even.

(e) If f, g are increasing , then $x_1 < x_2$ $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$.

Obviously, $f(x_1) + g(x_1) < f(x_2) + g(x_2)$. $\therefore$ $f + g$ is increasing.

fg is not increasing, $f(x) = x$, $g(x) = x^3$ are increasing, but $f(x)g(x) = x^4$ is not increasing.

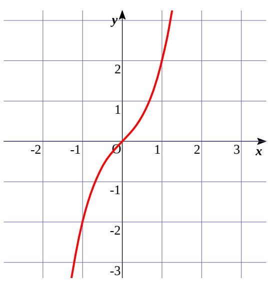
f **o** g is increasing since $x_1 < x_2 \Rightarrow g(x_1) < g(x_2) \Rightarrow f(g(x_1)) < f(g(x_2))$.

9.

(a) $y = x^3 + x$,

$y' = 3x^2 + 1$, $y'' = 6x$

Intercept & Inflexion: (0,0)



(b) $y = \dfrac{6x}{x^2+3}$

$y' = -\dfrac{x^2-3}{(x^2+3)^2}$

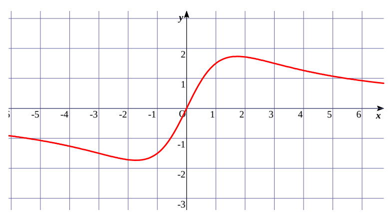
$y'' = \dfrac{12x(x+3)(x-3)}{(x^2+3)^3}$

Intercept : (0,0)

Max : $(\sqrt{3}, \sqrt{3})$, Min: $(-\sqrt{3}, -\sqrt{3})$

Inflexion :(0,0), Asymptote : y = 0

Symmetric about origin.



(c) $y = \dfrac{x-1}{x^2}$

$y' = \dfrac{2-x}{x^3}$, $y'' = \dfrac{2(x-3)}{x^4}$

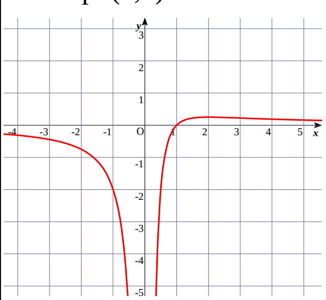
Max (2, 1/4)

Asymptotes :

x = 0 and y = 0

Inflexion (3, 2/9)

Intercept (1,0)



(d) $y = x + 3 + \dfrac{4}{x-1}$

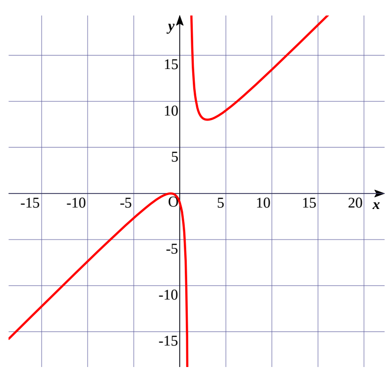
$y' = \dfrac{(x+1)(x-3)}{(x-1)^2}$

$y'' = \dfrac{8}{(x-1)^3}$

Intercept (-1,0)

Max (-1,0), Min (3, 8)

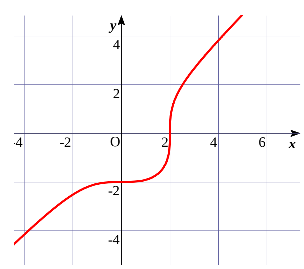
Asymptotes : x = 1, Oblique asymptote : y = x + 3



(e) $x^3 - y^3 = 2^3$

Intercept (2,0),(0,-2)

Asymptote: y = x



(f) $y = \dfrac{x^3}{x^2-1}$

$y' = \dfrac{x^2(x^2-3)}{(x-1)^2(x+1)^2}$

$y'' = \dfrac{2x(x^2+3)}{(x-1)^3(x+1)^3}$

Intercept:(0,0)

Symmetric about origin

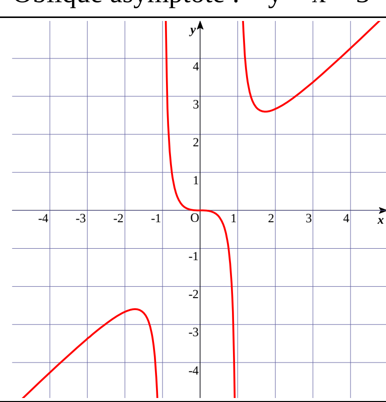
Max = $\left(-\sqrt{3}, -\dfrac{3}{2}\sqrt{3}\right)$

Min = $\left(\sqrt{3}, \dfrac{3}{2}\sqrt{3}\right)$

Inflexion : (0,0)

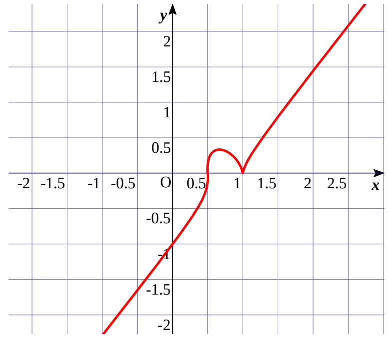
Asymptotes:

$x = \pm 1$, $y = x$



10. $y = (2x-1)^{1/3}(x-1)^{2/3}$ $\quad y' = \dfrac{2(3x-2)}{3(x-1)^{1/3}(2x-1)^{2/3}}$ $\quad y'' = -\dfrac{2}{9(x-1)^{4/3}(2x-1)^{5/3}}$

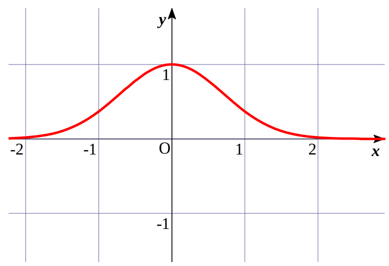
x	(-∞,1/2)	1/2	(1/2, 2/3)	2/3	(2/3,1)	1	(1, +∞)
y'	+	+∞	+	0	-	±∞	+
y"	-	±∞	+	+	+	+∞	+



Max : $\left(\dfrac{2}{3}, \dfrac{1}{3}\right)$ Min : (1 , 0)

Point of inflexion : $\left(\dfrac{1}{2}, 0\right)$

11. The required area $A = |x|e^{-x^2} = \begin{cases} xe^{-x^2} & , x \ge 0 \\ -xe^{-x^2} & , x < 0 \end{cases}$



$A' = \mp e^{-x^2}(2x^2 - 1) = 0 \quad \Rightarrow x = \pm\sqrt{\dfrac{1}{2}}$

$A'' = \pm 2e^{-x^2}x(2x^2 - 3) \quad \Rightarrow A''\Big|_{x = \pm\sqrt{\frac{1}{2}}} < 0$

Max of $A = \sqrt{\dfrac{1}{2}}\, e^{-\frac{1}{2}}$ at $x = \pm\sqrt{\dfrac{1}{2}}$.

The required distance $D = \sqrt{x^2 + e^{-2x^2}}$ (due to symmetry, we take $x > 0$ for simplicity)

$D' = \dfrac{e^{-2x^2}x(e^{2x^2} - 2)}{\sqrt{x^2 + e^{-2x^2}}} = 0 \quad \Rightarrow e^{2x^2} = 2, \quad x^2 = \dfrac{\ln 2}{2}, \quad x = \sqrt{\dfrac{\ln 2}{2}}$

When $x = \sqrt{\dfrac{\ln 2}{2}}$, $D = \sqrt{\dfrac{\ln 2 + 1}{2}}$.

When $0 < x < \sqrt{\dfrac{\ln 2}{2}}$, D' < 0 and when $x > \sqrt{\dfrac{\ln 2}{2}}$, D'> 0. ∴ Min of $D = \sqrt{\dfrac{\ln 2 + 1}{2}}$

12. $y = e^{-bt\cot\alpha}\sin bt$, $\quad y' = b\,e^{-bt\cot\alpha}\cos bt - b\,e^{-bt\cot\alpha}\cot\alpha\sin bt$

$\therefore \quad \dfrac{dy}{dt} = b\csc\alpha\; e^{-bt\cot\alpha}(\sin\alpha\cos bt - \cos\alpha\sin bt) = -b\csc\alpha\; e^{-bt\cot\alpha}\sin(bt-\alpha) = b\csc\alpha\; e^{-bt\cot\alpha}\sin(bt+\pi-\alpha)$

$\dfrac{d^2y}{dt^2} = b\csc\alpha\left[be^{-bt\cot\alpha}\cos(bt+\pi-\alpha) - b\cot\alpha e^{-bt\cot\alpha}\sin(bt+\pi-\alpha)\right]$

$= (b\csc\alpha)^2 e^{-bt\cot\alpha}[\sin\alpha\cos(bt+\pi-\alpha) - \cos\alpha\sin(bt+\pi-\alpha)]$

$= -(b\csc\alpha)^2 e^{-bt\cot\alpha}\sin(bt+\pi-2\alpha) = (b\csc\alpha)^2 e^{-bt\cot\alpha}\sin(bt+2\pi-2\alpha)$

y' = 0 $\Rightarrow \sin(bt + \pi - \alpha) = 0 \quad \Rightarrow \quad bt + \pi - \alpha = n\pi \quad \Rightarrow \quad t = \dfrac{(n-1)\pi + \alpha}{b}$ where $n \in \mathbf{Z}$.

(i) When $n = 2m - 1$ (odd), $m \in \mathbf{Z}$, $$t_{2m-1} = \frac{(2m-2)\pi + \alpha}{b} \qquad \text{.....(1)}$$

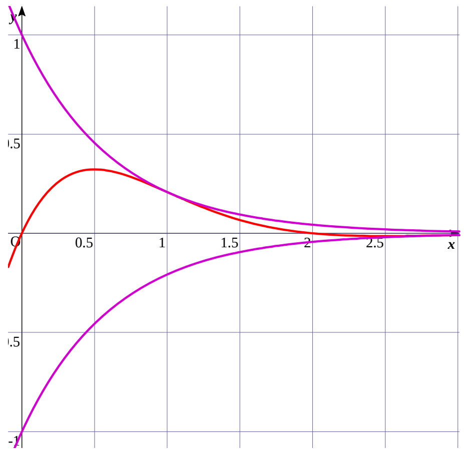
$$\frac{d^2y}{dt^2} = (b\csc\alpha)^2 e^{-bt_{2m-1}\cot\alpha}\sin(2m\pi - \alpha) = -(b\csc\alpha)^2 e^{-bt_{2m-1}\cot\alpha}\sin\alpha < 0$$

$\therefore$ y is max when $t = t_{2m-1}$ as in (1).

(ii) When $n = 2m$ (even), $m \in \mathbf{Z}$, $$t_{2m} = \frac{(2m-1)\pi + \alpha}{b} \qquad \text{.....(2)}$$

$$\frac{d^2y}{dt^2} = (b\csc\alpha)^2 e^{-bt_{2m}\cot\alpha}\sin(2m\pi + \pi - \alpha) = (b\csc\alpha)^2 e^{-bt_{2m}\cot\alpha}\sin(\pi - \alpha) = (b\csc\alpha)^2 e^{-bt_{2m}\cot\alpha}\sin\alpha > 0$$

$\therefore$ y is min when $t = t_{2m}$, as in (2).



Substitute the values of t_{2m-1} in $y = e^{-bt\cot\alpha}\sin bt$, we get :

$$y_{max} = e^{-[(2m-2)\pi+\alpha]\cot\alpha}\sin[(2m-2)\pi + \alpha]$$
$$= e^{-(2m-2)\pi\cot\alpha}\left\{e^{-\alpha\cot\alpha}\sin\alpha\right\} \qquad \text{.....(3)}$$

Since $t \geq 0$, $y = e^{-x}$ is a decreasing function,

the largest maximum is $e^{-\alpha\cot\alpha}\sin\alpha$,

where $t = \alpha/b$ where $m = 1$ in (1).

The last part graph is shown on the right above. $y = \pm e^{-\frac{\pi}{2}t}$ are the envelopes which encloses the curve $y = e^{-bt\cot\alpha}\sin bt$.

13. (a) $y = ax - \sin bx$, $y' = a - b\cos bx$ ..... (1)

Since $-1 \leq \cos bx \leq 1$, therefore y' is always between $a - b$ and $a + b$.

(b) From (1), $y'' = b^2\sin bx = 0 \Rightarrow \sin bx = 0 \Rightarrow bx = n\pi$, where $n \in \mathbf{Z}$.

$\therefore$ $x = n\pi/b$, $y''' = b^3\cos bx$, y''' at $x = n\pi/b$ is $\pm b^3 \neq 0$

When $x = n\pi/b$, $y = an\pi/b$. $(n\pi/b, an\pi/b)$, $n \in \mathbf{Z}$ are the points of inflexion.

By substitution, points of inflexion $(n\pi/b, an\pi/b)$ obviously lie on $y = ax$.

(c) From (1), $y' = 0 \Rightarrow \cos bx = a/b$, $bx = 2n\pi \pm \cos^{-1}\frac{a}{b}$, $x = \frac{2n\pi}{b} \pm \frac{1}{b}\cos^{-1}\frac{a}{b}$, $b \neq 0$.

$a, b \in +\mathbf{R}$, $a < b$, $0 < \cos^{-1}\frac{a}{b} < \frac{\pi}{2}$, $0 < \frac{1}{b}\cos^{-1}\frac{a}{b} < \frac{\pi}{2b}$

For $x = \frac{2n\pi}{b} + \frac{1}{b}\cos^{-1}\frac{a}{b}$, $bx = 2n\pi + \cos^{-1}\frac{a}{b}$, $\sin bx = \sqrt{1 - \frac{a^2}{b^2}}$, $y'' = b^2\sqrt{1 - \frac{a^2}{b^2}} > 0$

$\left(\frac{2n\pi}{b} + \frac{1}{b}\cos^{-1}\frac{a}{b}, a\left(\frac{2n\pi}{b} + \frac{1}{b}\cos^{-1}\frac{a}{b}\right) - \sqrt{1 - \frac{a^2}{b^2}}\right)$ are min. points lying on $y = ax - \sqrt{1 - \frac{a^2}{b^2}}$.

For $x = \frac{2n\pi}{b} - \frac{1}{b}\cos^{-1}\frac{a}{b}$, $bx = 2n\pi - \cos^{-1}\frac{a}{b}$, $\sin bx = -\sqrt{1 - \frac{a^2}{b^2}}$, $y'' = -b^2\sqrt{1 - \frac{a^2}{b^2}} < 0$

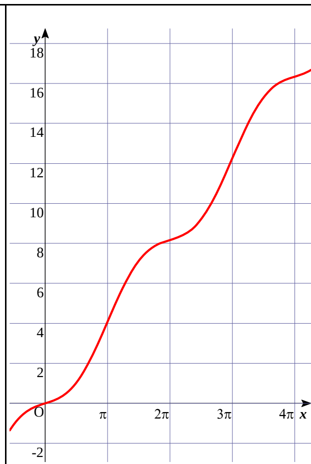
$\left(\frac{2n\pi}{b} - \frac{1}{b}\cos^{-1}\frac{a}{b}, a\left(\frac{2n\pi}{b} - \frac{1}{b}\cos^{-1}\frac{a}{b}\right) + \sqrt{1 - \frac{a^2}{b^2}}\right)$ are max. points lying on $y = ax + \sqrt{1 - \frac{a^2}{b^2}}$.

(d)

(i) For $a > b$,

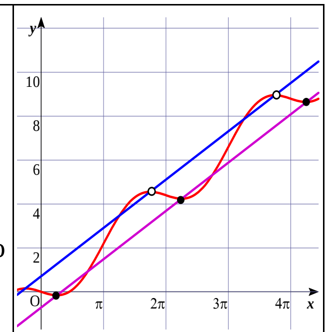
there is no max. and min. points.

The graph on the right with $a = 1.3$ and $b = 1$.



(ii) For $a < b$,

The graph on the right with $a = 0.7$ and $b = 1$. The lines $y = ax \pm \sqrt{1 - \frac{a^2}{b^2}}$ are also drawn with max. (in hollow dots) and min. (in solid dots) shown.



14. $y^2 = \dfrac{a^2(x-a)}{x}$. When $x \to 0$, $y \to \infty$, therefore $x = 0$ is a vertical asymptote.

$$\lim_{x\to\infty}\left(\frac{y}{x}\right)^2 = \lim_{x\to\infty}\frac{a^2(x-a)}{x^3} = 0 \qquad \Rightarrow m = \lim_{x\to\infty}\frac{y}{x} = 0$$

$$\lim_{x\to\infty}(y - mx) = \lim_{x\to\infty} y = \pm\sqrt{\lim_{x\to\infty} y^2} = \pm\sqrt{\lim_{x\to\infty}\frac{a^2(x-a)}{x}} = \pm\sqrt{\lim_{x\to\infty} a^2\left(1-\frac{a}{x}\right)} = \pm\sqrt{a^2} = \pm a$$

$\therefore$ $y = \pm a$ are oblique aymptotes.

Let (h, k) be a point on the given curve. Differentiate the function, we have

$$\frac{dy}{dx} = \frac{a^3}{2yx^2} \qquad \Rightarrow \left.\frac{dy}{dx}\right|_{(h,k)} = \frac{a^3}{2kh^2}$$

Let $y = mx$ be the equation of tangent which passes through the origin.

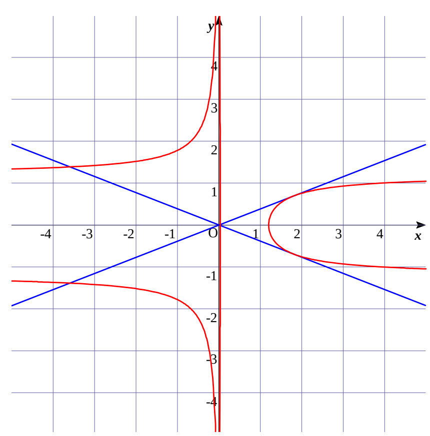
$$m = \frac{k}{h} = \frac{a^3}{2kh^2} \Rightarrow k^2 = \frac{a^3}{2h} \qquad \text{….(1)}$$

But (h, k) is on the curve, we have $k^2 = \dfrac{a^2(h-a)}{h}$ ….(2)

Solve (1) and (2), $(h, k) = \left(\dfrac{3a}{2}, \pm\dfrac{a}{3}\sqrt{3}\right)$ and the required tangents are $y = \pm\dfrac{2}{9}\sqrt{3}\,x$, or $y^2 = \dfrac{4}{27}x^2$ …(3)

$$\lambda x^3 = a^2(x-a) \quad \Leftrightarrow \quad \lambda x^2 = \frac{a^2(x-a)}{x} \Leftrightarrow \begin{cases} y^2 = \lambda x^2 & (4) \\ y^2 = \dfrac{a^2(x-a)}{x} & (5) \end{cases}$$

Since (5) is the given curve and in order that $\lambda x^3 = a^2(x-a)$ has three real roots as in the graph on the right, $0 < \lambda < 4/27$, by comparing (3) and (4). It has three roots ($x = -3a, 3a/2, 3a/2$) if $\lambda = 4/27$ and has only one root if λ does not lie between these limits.



15. $x = 3t^2$, $y = 2t^3$ $\Rightarrow$ $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2 \Rightarrow \frac{dy}{dx} = t$

Let the tangent(s) to the curve pass through a given point (x_1, y_1) be $y - y_1 = t(x - x_1)$.

$x = 3t^2$, $y = 2t^3$ is on this tangent. $\therefore$ $2t^3 - y_1 = t(3t^2 - x_1)$ or $t^3 - x_1t - y_1 = 0$ ....(1)

(1) is a cubic equation in t and in general has three roots. Therefore in general, three tangents to the curve pass through a given point (x_1, y_1).

(i) Consider the function $y = t^3 - x_1t - y_1$ ....(2)

$$y' = 3t^2 - x_1 = 0 \Rightarrow t = \pm\sqrt{\frac{x_1}{3}} \quad \text{....(3)}$$

$\therefore$ (3) has a real solution if $x_1 > 0$, in other words, (2) has turning point(s) if $x_1 > 0$.

$\therefore$ Since the curve (2) cuts the x-axis once if there is no turning point and therefore $x_1 > 0$ is a necessary condition for the three tangents to be real.

(ii) The cubic equation (1) has distinct real roots if and only if $\frac{x_1^3}{27} - \frac{y_1^2}{4} > 0$, which is the sufficient conditions for the three tangents to be real and distinct.

16. $x_0 = a\cos^2 t \sin t$, $y_0 = a \cos t \sin^2 t$.

$$\frac{dx}{dt} = a\cos t[\cos^2 t - 2\sin^2 t], \quad \frac{dy}{dt} = a\sin t[2\cos^2 t - \sin^2 t], \quad \frac{dy}{dx} = \frac{\tan t[2\cos^2 t - \sin^2 t]}{\cos^2 t - 2\sin^2 t}$$

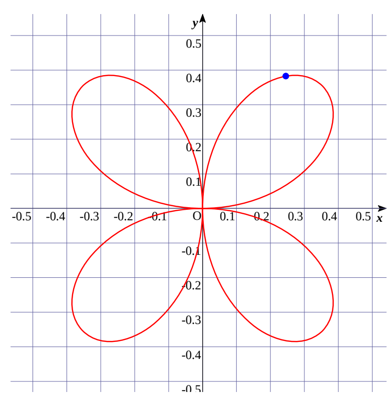
Equation of tangent is given by: $y - a\cos t \sin^2 t = \frac{\tan t[2\cos^2 t - \sin^2 t]}{\cos^2 t - 2\sin^2 t}(x - a\cos^2 t \sin t)$

At $y = 0$, $x_1 = a\cos^2 t \sin t - \frac{a\cos t \sin^2 t(\cos^2 t - 2\sin^2 t)}{\tan t(2\cos^2 t - \sin^2 t)} = \frac{a\cos^2 t \sin t}{3\cos^2 t - 1}$ ....(1)

$$x_1\left(x_1 - \frac{x_0}{2}\right) = \frac{a\cos^2 t \sin t}{3\cos^2 t - 1}\left[\frac{a\cos^2 t \sin t}{3\cos^2 t - 1} - \frac{a\cos^2 t \sin t}{2}\right] = \frac{1}{2}\left(\frac{a\cos^2 t \sin t}{3\cos^2 t - 1}\right)^2[2 - (3\cos^2 t - 1)]$$

$$= \frac{1}{2}\left(\frac{a\cos^2 t \sin t}{3\cos^2 t - 1}\right)^2(3\sin t)^2 \ge 0 \Rightarrow \begin{cases} x_1 \ge 0 \\ x_1 \ge x_0/2 \end{cases} \text{ or } \begin{cases} x_1 \le 0 \\ x_1 \le x_0/2 \end{cases}$$

In either cases, x_1 does not lie between 0 and $\frac{x_0}{2}$.



$$\text{Area} = \frac{1}{2}\int_0^{\pi/2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/2}(a^2\cos^2 t \sin^2 t)dt = \frac{a^2}{8}\int_0^{\pi/2}(\sin 2t)^2 dt$$

$$= \frac{a^2}{16}\int_0^{\pi/2}(1 - \sin 8t)dt = \frac{a^2}{16}\left[t + \frac{\cos 8t}{8}\right]_0^{\pi/2} = \underline{\underline{\frac{\pi a^2}{32}}}$$

17. $y = \frac{e^{ax}}{1 + x^2}$, $\frac{dy}{dx} = \frac{e^{ax}(ax^2 - 2x + a)}{(1 + x^2)^2} = 0 \Rightarrow ax^2 - 2x + a = 0$

The function has turning point if $ax^2 - 2x + a = 0$ has solution, that is,

$\Delta = (-2)^2 - 4(a)(a) = 4(1 - a^2) \ge 0$ or $|a| \le 1$.

If $a = 1$, $\dfrac{dy}{dx} = \dfrac{e^{ax}(x^2 - 2x + 1)}{(1+x^2)^2} = \dfrac{e^{ax}(x-1)^2}{(1+x^2)^2}$, there is no sign change about any point on the curve.

If $a = -1$, $\dfrac{dy}{dx} = \dfrac{e^{ax}(-x^2 - 2x - 1)}{(1+x^2)^2} = -\dfrac{e^{ax}(x+1)^2}{(1+x^2)^2}$, there is no sign change about any point on the curve.

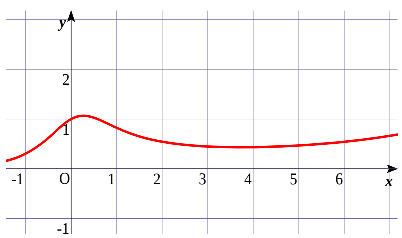
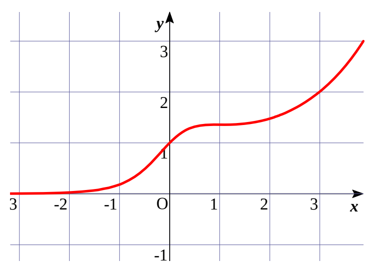
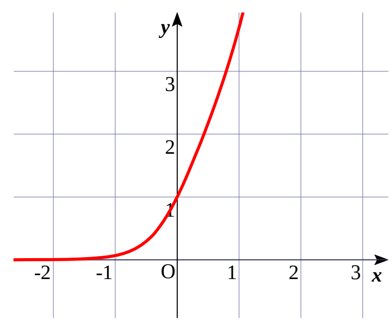
In both cases, the function has no max. or min.

For $|a| < 1$, let $ax^2 - 2x + a = a(x - \alpha)(x - \beta)$ $\alpha < \beta$.

$\dfrac{dy}{dx} = \dfrac{e^{ax}a(x-\alpha)(x-\beta)}{(1+x^2)^2}$, and there are sign changes about $x = \alpha$ and $x = \beta$.

In conclusion, y has a maximum or a minimum value if $|a| < 1$.

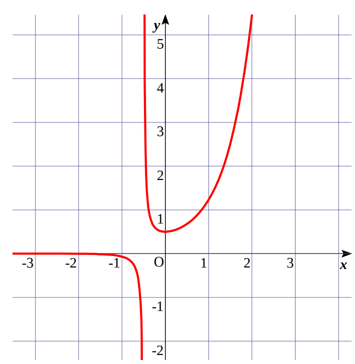
Obviously, $ax^2 - 2x + a = 0$ has no solution if $\Delta < 0$ or $|a| > 1$.

a = 1/2	a = 1	a = 2
		

18. **(i)** $y = \dfrac{e^{2x}}{a+bx}$, $\dfrac{dy}{dx} = \dfrac{e^{2x}(2bx + 2a - b)}{(a+bx)^2} = 0$ $\Rightarrow x = \dfrac{b-2a}{2b}, y = \dfrac{2e^{(b-2a)/b}}{b}$

For the given stationary value we have : $0 = \dfrac{b-2a}{2b}, \dfrac{1}{2} = \dfrac{2e^{(b-2a)/b}}{b}$ $\Rightarrow$ $a = 2, b = 4$.

For these values of a, b $y = \dfrac{e^{2x}}{2+4x}$, $\dfrac{dy}{dx} = \dfrac{e^{2x}(8x)}{(2+4x)^2}$



When x is slightly less than 0, $y' < 0$.

When x is slightly greater than 0, $y' > 0$.

$\therefore$ $(0, 1/2)$ is a minimum point.

(ii) $y = e^{ax}$, $y' = ae^{ax}$

At the point (h, k) on the curve, $k = e^{ah}$, $y' = ae^{ah}$.

The gradient from $(0, 0)$ to (h, k) is k/h. $\therefore$ $k/h = a\,e^{ah} = ak$

$\therefore$ $h = 1/a$ or $k = 0$.

When $k = 0$, $e^{ah} = 0$, $h = -\infty$ $(a > 0)$ or $h = +\infty$ $(a < 0)$ (rejected)

When $h = 1/a$, $k = e^{ah} = e$, $(h, k) = (1/a, e)$ $m = ae^{ax} = ae$

Eq. of tangent : $y - e = ae(x - 1/a)$ or $y = aex$.

$kx = e^{ax}$ $\Leftrightarrow$ $\begin{cases} y = e^{ax} & (1) \\ y = kx & (2) \end{cases}$ which has no root if $\begin{cases} 0 < k < ae & \text{if } a > 0 \\ ae < k < 0 & \text{if } a < 0 \end{cases}$.

19. $f(x)=\dfrac{(x-1)^3}{(x+1)^2}$ (1) $\qquad f'(x)=\dfrac{(x+5)(x-1)^2}{(x+1)^3}$ (2) $\qquad f''(x)=\dfrac{24(x-1)}{(x+1)^4}$ (3)

(a) When $x\to -1$, $y\to\infty$, therefore $x = -1$ is a vertical asymptote.

$$m=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{(x-1)^3}{x(x+1)^2}=\lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^3}{\left(1+\frac{1}{x}\right)^2}=1,$$

$$c=\lim_{x\to\infty}\left(\frac{(x-1)^3}{(x+1)^2}-x\right)=\lim_{x\to\infty}\left(\frac{(x-1)^3-x(x+1)^2}{(x+1)^2}\right)=\lim_{x\to\infty}\frac{-5x^2+2x-1}{(x+1)^2}=\lim_{x\to\infty}\frac{-5+\frac{2}{x}-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)^2}=-5$$

$\therefore$ $y = x - 5$ is an oblique asymptote of the graph $y = f(x)$.

(b) **(i)** When $x = -5$, $f'(x) = 0$, $f''(x) = -9/16 < 0$. $(-5, -13.5)$ is a local max. point.

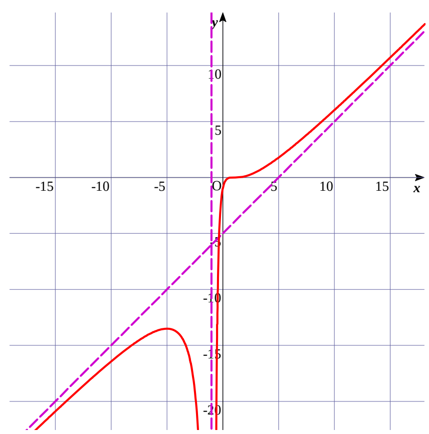
When $x = 1$, $f''(x) = 0$ and changes sign across the value, $(1, 0)$ is a point of inflexion.

(ii) Substitute $y = x - 5$ in (1), $(x - 5)(x + 1)^2 = (x - 1)^3$, $x = -1/3$.

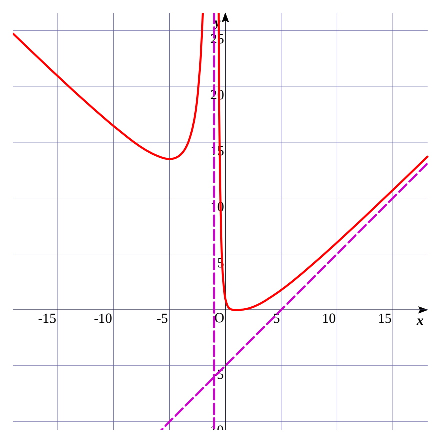
$\left(-\frac{1}{3},-\frac{16}{3}\right)$ is the the intersection of the graph

$y = f(x)$ and its oblique asymptote.

(c)



(d)



20. **(i)** Counterexample : $f(x)=\begin{cases}x & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases}$. Then $f(x) \ge f(y)$ if $x \ge y$. But $f'(x)$ is not defined at $x = 0$.

(ii) True. Proof: $f'(x)=\lim_{x\to h}\dfrac{f(x)-f(h)}{x-h}>0$, since $f(x) - f(h)$ and $x - h$ are of the same sign.

(iii) True. Proof: $f'(x) > 0$ for all x $\Rightarrow$ $\dfrac{f(x)-f(y)}{x-y}=f'(\xi)>0$, $y<\xi<x$, by Mean Value Theorem.

$\Rightarrow$ $f(x) > f(y)$ whenever $x > y$.

(iv) True. Proof: $f'(x) \ge 0$ for all x $\Rightarrow$ $\dfrac{f(x)-f(y)}{x-y}=f'(\xi)\ge 0$, $y<\xi<x$, by Mean Value Theorem.

$\Rightarrow$ $f(x) \ge f(y)$ whenever $x > y$.